J. Fluid Mech. (1991), *vol.* 231, *pp.* 51–71
Printed in Great Britain

Slow viscous flow due to motion of an annular disk; pressure-driven extrusion through an annular hole in a wall

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(Received 27 February 1990 and in revised form 12 October 1990)

The description of the slow viscous flow due to the axisymmetric or asymmetric translation of an annular disk involves the solution of respectively one or two sets of triple integral equations involving Bessel functions. An efficient method is presented for transforming each set into a Fredholm integral equation of the second kind. Simple, regular kernels are obtained and the required physical constants are readily available. The method is also applied to the pressure-driven extrusion flow through an annular hole in a wall. The velocity profiles in the holes are found to be flatter than expected with correspondingly sharper variation near a rim. For the sideways motion of a disk, an exact solution is given with bounded velocities and both components of the rim pressure singularity minimized. The additional drag experienced by this disk when the fluid is bounded by walls parallel to the motion is then determined by solving a pair of integral equations, according to methods given in an earlier paper.

1. Introduction

Although the classical Stokes flow problem of the motion of an inertialess unbounded fluid past a single axisymmetric body has been studied for more than 100 years, analytical solutions have been obtained only for special geometries in which the bodies usually correspond to a complete coordinate surface of one of the special orthogonal coordinate systems in which the Stokes equations are simply separable. The exception involving an incomplete surface is the spherical cap for which uniqueness was guaranteed by minimizing the stress singularity at the rim or, equivalently, imposing a continuous velocity field in each of the different solutions given by Collins (1963) and Dorrepaal, O'Neill & Ranger (1976) for axisymmetric flow and in the asymmetric flow solution constructed by Dorrepaal (1976). The exception involving a non-convex body is the torus for which solutions were constructed in terms of the stream function by Payne & Pell (1960) and directly in terms of the velocity components by Majumdar & O'Neill (1977) for axisymmetric flow and by Goren & O'Neill (1980) for asymmetric flow. When the stream function is used in conjunction with toroidal coordinates, there are coordinate surfaces that enclose the body and the unknown flux constant must be determined by requiring the pressure to be continuous. However, when the stream function is the subject of a mixed boundary-value problem, the pressure continuity conditions ensure single-valuedness but the solution will in general contain terms that yield unbounded velocities at the rim. The need to remove such terms by adding eigenfunctions to the stream function was discussed by Hasimoto (1981). This strategy was further developed by Miyazaki & Hasimoto (1984) for the flow due to an arbitrarily placed stokeslet in the presence of a hole in a wall in first finding particular solutions for the potential functions and then adding complementary harmonic functions to remove the rim singularities in the velocity fields. Similar features can occur in the application of the Wiener–Hopf technique.

The Stokes drag on a flat annular ring was studied experimentally by Roger & Hussey (1982) who also presented a theoretical calculation using a beads-on-a-shell model to represent a distribution of point forces. The axisymmetric streaming flow past an annular disk can be solved by using a distribution of ringlets to obtain the same integral equation as in the corresponding electrostatic potential problem of Cooke (1963). Unfortunately this equation is singular at the inner rim and the only way in which the method can allow the removal of the singularity is by transferring it to the outer rim. Despite the consequent numerical inefficiency, both computations achieved good accuracy for values of the drag force. Asymptotic estimates for the limit in which the radii difference approaches zero were given by Spence (1970). This limit is of particular interest because, as described by Stewartson (1983) for the thin torus, the viscous fluid exhibits a marked reluctance to flow through the boundary and the flux function has an essential singularity in the limit in which the hollow boundary approaches a circle and disappears. This situation was also demonstrated in calculations by Price (1985) for a finite circular pipe and by Davis (1985) for a spherical shell with axisymmetric caps removed in which the boundary conditions lead to sets of triple integrals or triple series that can at best be reduced to integral equations of the second kind, not necessarily disjoint. Lasso & Weidman (1986) extrapolated their experimental results for finite hollow cylinders of non-zero thickness to show good agreement with Price's results.

In considering principally the annular disk, this paper shows, in §2, that the singularity, encountered by earlier authors in working with the unknown pressure jump across the disk, can be avoided by introducing two functions that determine the unknown normal velocity in the complementary intervals. The triple integral equations, which arise from the boundary conditions, are then transformed, by elementary Hankel transforms, into a pair of equations that reduce to a single Fredholm integral equation of the second kind with a simple, bounded kernel. This method is considerably more efficient than that described by Sneddon (1966). No arbitrary constants appear here because the symmetry of the flow allows use of a representation in terms of a harmonic function equal to the normal velocity in the plane of the disk. Section 3 considers a viscous flow past the complementary boundary, namely the pressure-driven extrusion of fluid through an annular hole in a plane wall. This flow is regarded as a modification of that described by Sampson (1891) for the circular hole and is determined by using distributions of ringlet singularities on the wall and central disk. The analysis is similar to that above and in both cases the dimensionless flux and drag coefficients are readily determined from the numerical solution of each integral equation. Moreover, the velocity profiles in the holes are found to exhibit an interesting feature; they increasingly flatten, as the hole is enlarged, with correspondingly increased gradients near the rims, where square-root behaviour is exhibited.

Asymmetric motions are studied by means of the velocity representation used by Ranger (1978). An exact solution is available for the disk in unbounded fluid and the two constants necessarily introduced are determined by requiring both tangential velocity components to be bounded. The velocity components and pressure are evaluated in terms of toroidal coordinates to obtain simple formulae similar to those for the shear flow disturbance due to a hole in the wall (Davis 1991*b*). Section 4 is continued by supposing that this sideways motion of the disk occurs midway between parallel walls. The influence of reflected velocity fields on the tangential drag force is determined by a pair of integral equations obtained by the methods described by Davis (1990) for axisymmetric disk motions in the presence of fixed boundaries. Section 5 discusses the sideways motion of an annular disk in unbounded fluid. In this case, four constants are reduced to three determined by consistency conditions that arise after imposing a continuous velocity field. For a given ratio of radii, the tangential drag force is smaller than that obtained by Goren & O'Neill (1980) but its fractional variation with the radii ratio, i.e. variation of the dimensionless drag coefficient, is remarkably similar to that of the normal drag force in axisymmetric translation.

2. The sedimenting annular disk

A thin rigid annular disk of radii $a, 1$ $(a < 1)$ translates steadily along its axis with speed U in incompressible viscous fluid that is at rest at infinity. Cylindrical polar coordinates (ρ, θ, z) are chosen so that the disk is instantaneously at $z = 0$ $(a \leqslant \rho \leqslant 1, 0 \leqslant \theta < 2\pi)$ and moving towards the half-space $z > 0$. The Reynolds number is assumed to be sufficiently small for the velocity field $\boldsymbol{v}$ to satisfy the creeping-flow equations

$$\mu \nabla^2 \boldsymbol{v} = \nabla p, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2.2}$$

where μ is the coefficient of viscosity and p the dynamic pressure. The boundary conditions on the disk are

$$v_\rho = 0, \quad v_z = U \quad \text{on} \quad z = 0 \quad (a \leqslant \rho \leqslant 1). \tag{2.3}$$

For this geometry, with the flow symmetric about the plane $z = 0$, an appropriate representation of the velocity field is given by

$$\boldsymbol{v} = \phi \hat{\boldsymbol{z}} - z \nabla \phi, \quad p = -2\mu \frac{\partial \phi}{\partial z}, \tag{2.4}$$

where

$$\nabla^2 \phi = 0. \tag{2.5}$$

The axisymmetric function $\phi(\rho, z)$ has the form

$$\phi = \frac{2U}{\pi} \int_0^\infty A(k)\, J_0(k\rho)\, \mathrm{e}^{-k|z|}\, \mathrm{d}k, \tag{2.6}$$

whence, from (2.4), the normal velocity and pressure jump are given by

$$v_z(\rho, 0) = \frac{2U}{\pi} \int_0^\infty J_0(k\rho)\, A(k)\, \mathrm{d}k, \tag{2.7}$$

$$[p]_{z=0} = \frac{8U\mu}{\pi} \int_0^\infty J_0(k\rho)\, kA(k)\, \mathrm{d}k. \tag{2.8}$$

The condition $v_z(\rho, 0) = U$ $(a \leqslant \rho \leqslant 1)$ in (2.3) and the pressure continuity requirement $[p]_{z=0} = 0$ for $0 \leqslant \rho < a$ and $\rho > 1$ now yield triple integral equations

$$\int_0^\infty A(k)\, J_0(k\rho)\, \mathrm{d}k = \tfrac{1}{2}\pi \quad (a \leqslant \rho \leqslant 1), \tag{2.9}$$

$$\int_0^\infty kA(k)\, J_0(k\rho)\, \mathrm{d}k = 0 \quad (0 \leqslant \rho < a, \rho > 1). \tag{2.10}$$

These can be solved by the methods described by Sneddon (1966, §6.2) but a simpler procedure for determining the physical quantities of interest is available.

First, the Fourier transforms of $J_0(k\rho)$ and the consequent identity

$$\int_0^\infty J_0(k\rho)\frac{\sin k}{k}\,\mathrm{d}k = \begin{cases}\tfrac{1}{2}\pi & (0 \leqslant \rho \leqslant 1)\\ \sin^{-1}(\rho^{-1}) & (\rho > 1)\end{cases}, \tag{2.11}$$

show that the disk condition (2.9) is satisfied by writing

$$A(k) = \frac{\sin k}{k} - \int_0^a t^{-1}F(t)\sin kt\,\mathrm{d}t - \int_1^\infty G(t)\cos kt\,\mathrm{d}t. \tag{2.12}$$

Meanwhile, the identities

$$\int_u^\infty \frac{\rho J_0(k\rho)}{(\rho^2-u^2)^{\frac{1}{2}}}\mathrm{d}\rho = \frac{\cos ku}{k}, \quad \int_0^u \frac{\rho J_0(k\rho)}{(u^2-\rho^2)^{\frac{1}{2}}}\mathrm{d}\rho = \frac{\sin ku}{k} \tag{2.13}$$

enable the pressure continuity conditions (2.10) to be simplified to the forms

$$\int_0^\infty A(k)\cos ku\,\mathrm{d}k = 0 \quad (u \geqslant 1), \qquad \int_0^\infty A(k)\sin ku\,\mathrm{d}k = 0 \quad (u \leqslant a),$$

which, on substitution of (2.12), yield

$$\int_0^a \frac{F(t)}{u^2-t^2}\mathrm{d}t = \frac{\pi}{2}G(u) \quad (u \geqslant 1), \tag{2.14}$$

$$\tfrac{1}{2}\ln\left(\frac{1+u}{1-u}\right) = \frac{\pi}{2}\frac{F(u)}{u} - u\int_1^\infty \frac{G(t)}{t^2-u^2}\mathrm{d}t \quad (0 \leqslant u \leqslant a).$$

Thus $F(t)$ satisfies the integral equation of the second kind

$$F(u) - \frac{2u^2}{\pi^2}\int_0^a\left[\frac{1}{u}\ln\left(\frac{1+u}{1-u}\right) - \frac{1}{v}\ln\left(\frac{1+v}{1-v}\right)\right]\frac{F(v)}{u^2-v^2}\mathrm{d}v = \frac{u}{\pi}\ln\left(\frac{1+u}{1-u}\right) \quad (0 \leqslant u \leqslant a). \tag{2.15}$$

The normal velocity in the hole is given, from (2.7), (2.11), and (2.12) by

$$v_z(\rho, 0) = U - \frac{2U}{\pi}\int_\rho^a \frac{t^{-1}F(t)}{(t^2-\rho^2)^{\frac{1}{2}}}\mathrm{d}t \quad (0 \leqslant \rho < a) \tag{2.16}$$

and hence the relative flux of fluid, πUM, through the hole in the disk in the negative z-direction is determined by

$$M = \frac{4}{\pi}\int_0^a F(t)\,\mathrm{d}t. \tag{2.17}$$

The drag force $16U\mu D$ exerted on the disk by the fluid is determined, from (2.8), by

$$\begin{aligned} D &= \int_0^\infty A(k)\,[J_1(k) - aJ_1(ka)]\,\mathrm{d}k \\ &= (1-a^2)^{\frac{1}{2}} + \int_0^a F(t)\,[(a^2-t^2)^{-\frac{1}{2}} - (1-t^2)^{-\frac{1}{2}}]\,\mathrm{d}t + \int_1^\infty G(t)\,[(t^2-1)^{-\frac{1}{2}} - (t^2-a^2)^{-\frac{1}{2}}]\,t\,\mathrm{d}t \\ &= (1-a^2)^{\frac{1}{2}} + \frac{2}{\pi}\int_0^a \frac{F(t)}{(a^2-t^2)^{\frac{1}{2}}}\sin^{-1}\left[\left(\frac{1-a^2}{1-t^2}\right)^{\frac{1}{2}}\right]\mathrm{d}t \end{aligned} \tag{2.18}$$

after substitution of (2.12) and (2.14). Lastly, the essential features of the relative velocity in the hole can be deduced by noting that, in (2.16),

$$\frac{2U}{\pi}\int_\rho^a \frac{t^{-1}F(t)}{(t^2-\rho^2)^{\frac{1}{2}}}\,\mathrm{d}t \sim \frac{2UF(a)}{\pi a^2}(a^2-\rho^2)^{\frac{1}{2}} \quad \text{as} \quad \rho \to a \tag{2.19}$$

and comparing the axial value with that of the Sampson profile (see (3.3)) having the same flux, namely $3UM\,(a^2-\rho^2)^{\frac{1}{2}}/2a^3$, where M is given by (2.17). Thus the quantity

$$\frac{2a^2}{3M}\int_0^a t^{-2}F(t)\,\mathrm{d}t,$$

which is convergent because the integral equation (2.15) yields

$$\lim_{t\to 0}\frac{F(t)}{t^2} = \frac{2}{\pi}-\frac{2}{\pi^2}\int_0^a\left[2-\frac{1}{v}\ln\left(\frac{1+v}{1-v}\right)\right]v^{-2}F(v)\,\mathrm{d}v,$$

determines whether the relative velocity profile is flatter or more peaked than the elliptical profile of the flow through a hole in a wall.

For $a \ll 1$, the solution of (2.15) is

$$F(u) \sim \frac{2u^2}{\pi}[1+O(a^3)]+\frac{2u^4}{3\pi},$$

whence substitution in (2.16), (2.17) and (2.18) shows that the normal velocity in the hole, the relative flux through the hole and the dimensionless drag coefficient are approximated by

$$v_z(\rho,0) \sim U-\frac{4U}{\pi^2}(a^2-\rho^2)^{\frac{1}{2}}, \quad \pi UM \sim \frac{8U}{3\pi}a^3,$$

$$D = (1-a^2)^{\frac{1}{2}}+\int_0^a \frac{F(t)}{(a^2-t^2)^{\frac{1}{2}}}\left[1-\frac{2}{\pi}\sin^{-1}\left(\frac{a^2-t^2}{1-t^2}\right)^{\frac{1}{2}}\right]\mathrm{d}t$$

$$\sim 1-\frac{4a^3}{3\pi^2}.$$

As mentioned by Roger & Hussey (1982), this last result can be obtained by an application of the reciprocal theorem (Happel & Brenner 1973). The relative velocity and flux correspond to a Sampson flow (see (3.3)) with pressure drop $8U\mu/\pi$ which is, indeed, the axial pressure jump for the translating disk without a hole. As a increases from zero towards unity, the relative velocity profile departs from the Sampson profile but retains the important feature (2.19).

Evidently M/a^3 is an appropriate measure of flux variation while, as noted above, the axial velocity in the hole must be scaled with $3UM/2a^2$ for comparison with the Sampson profile. The integral equation (2.15) is easily solved numerically by a Chebyshev polynomial approximation as described in the Appendix. Values of the dimensionless flux factor M/a^3, drag force D and axial relative velocity $2a^2\,[1-U^{-1}v_z(0,0)]/3M$ are listed in table 1. As a increases towards unity, M and D exhibit considerable reluctance to approach their limit values while the flattening of the velocity profile indicates a corresponding steepening near the inner rim. This phenomenon arises from the need to recover the uniform stream in the limit $a = 1$ and contrasts with the entrance profile for pressure-driven flow through an orifice, namely, the average of Sampson and Poiseuille flows having the same flux (Dagan,

a	ϵ^{-1}	M/a^3	D	$2a^2[1-U^{-1}v_z(0,0)]/3M$
1/2	1	0.2874	0.9810	0.9756
2/3	2	0.3064	0.9494	0.9525
4/5	4	0.3350	0.8977	0.9226
5/6	5	0.3459	0.8778	0.9123
8/9	8	0.3709	0.8329	0.8906
11/12	11	0.3890	0.8015	0.8763
24/25	24	0.4349	0.7259	0.8442
49/50	49	0.4762	0.6634	0.8194
99/100	99	0.5144	0.6094	0.7992

TABLE 1. Values of the dimensionless flux factor M/a^3, drag force D and relative velocity $2a^2[1-U^{-1}v_z(0,0)]/3M$ for various values of inner radius a of the annular disk

Weinbaum & Pfeffer 1982). Further evidence of this steepening of the profile near a rim will be presented in the next section. The computed values suggest that the empirical formulae

$$M \sim 1-\frac{4}{\ln(32/\epsilon)}, \quad D \sim \frac{\pi^2}{2\ln(32/\epsilon)} \quad \text{as} \quad \epsilon = a^{-1}-1 \to 0 \tag{2.20}$$

describe the approach of the dimensionless flux and drag to their limit values.

Similar formulae have been obtained for related flows. For a torus with radii ratio $\delta(\ll 1)$, Majumdar & O'Neill (1977) and Stewartson (1983) respectively established the asymptotic estimates $4\pi/3[\ln(8/\delta)+\frac{1}{2}]$ for the drag coefficient and $1-4/[\ln(8/\delta)+\frac{1}{2}]$ for the flux factor. For a finite pipe of unit radius and length $b\,(\ll 1)$, Price (1985) obtained the empirical formulae

$$(1-M)^{-1} \sim \tfrac{1}{4}[\ln(8\pi/b)+\tfrac{1}{2}] \sim 4\pi^2/D.$$

In the above analysis, equation (2.9) expresses, on substitution in (2.7), the normal velocity in terms of two functions $F(t)$, $G(t)$, defined on either side of the rims of the disk and then two equations are obtained by requiring the pressure to be continuous and hence single-valued. Evidently this solution is analytically and computationally better than that obtained by writing the pressure jump across the disk in the form

$$[p]_{z=0} = -\frac{8U\mu}{\pi\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^1 \frac{tS(t)}{(t^2-\rho^2)^{\frac{1}{2}}}\mathrm{d}t \quad (a<\rho<1)$$

and requiring the normal velocity at the disk to be U. This procedure is successful for the complete disk ($a=0$, Davis 1990) but in the annular case yields the integral equation

$$S(t) = \frac{t}{(t^2-a^2)^{\frac{1}{2}}}\left[1-\frac{4}{\pi^2}\int_a^1 K(t,u)\frac{uS(u)}{(u^2-a^2)^{\frac{1}{2}}}\mathrm{d}u\right] \quad (a<t\leqslant 1), \tag{2.21}$$

where $$K(t,u) = \frac{1}{2(t^2-u^2)}\left\{\frac{t^2-a^2}{t}\ln\left(\frac{t+a}{t-a}\right)-\frac{u^2-a^2}{u}\ln\left(\frac{u+a}{u-a}\right)\right\}.$$

The appearance of the singularity $(u-a)^{-1}$ in the kernel of this integral equation for $S(t)\,t(t^2-a^2)^{-\frac{1}{2}}$ suggests that two unknown functions are required to take proper account of both rim singularities. The numerical solution of (2.21), given by Cooke (1963) in the context of electrostatics, is handicapped by the logarithmic singularity

a	ϵ^{-1}	D	D (Cooke)	D (2.20)	D (2.22)
1/2	1	0.9810	0.9810	—	—
2/3	2	0.9494	0.9494	—	—
4/5	4	0.8977	0.8976	1.017	0.8838
5/6	5	0.8778	0.8776	0.9723	0.8684
8/9	8	0.8329	0.8326	0.8899	0.8288
11/12	11	0.8015	0.8011	0.8416	0.7991
24/25	24	0.7259	—	0.7428	0.7254
49/50	49	0.6634	0.6667	0.6707	0.6639
99/100	99	0.6094	—	0.6122	0.6087

TABLE 2. Comparison of computed values of D in table 1 with Cooke's (1963) numerics and the asymptotic formulae (2.20) and (2.22)

in $S(t)$ at the inner rim and consequently needs many more truncation points than in (2.15) to achieve comparable accuracy even for the two integrals M and D. Since these quantities change rapidly as $a \to 1$, Spence (1970) considered the asymptotic solution of (2.21) in this limit and established, by Wiener–Hopf methods, the drag coefficient estimate in (2.20). He then suggested, by comparison with Cooke's results, that the 2 factor in this formula be replaced empirically by $(2+\epsilon)$ and finally, in a note added in proof, gave the more accurate result

$$D \sim \pi^2/[(2+\epsilon)\ln(32/\epsilon)+\epsilon+O(\epsilon^2\ln\epsilon)]. \tag{2.22}$$

Table 2 displays, for comparison, values of D from table 1, Cooke's numerics and the asymptotic formulae (2.20) and (2.22). Spence's method has been applied by Davis (1991*a*) to annular disks moving axisymmetrically in the presence of fixed boundaries. These calculations were guided by a preference for integral equations of the second kind, for computational reasons described in detail by Karrila & Kim (1989). However, Leppington & Levine (1972) obtained the asymptotic formulae (2.20), (2.22) more directly by estimating the kernel of the corresponding integral equation of the first kind. Their analysis shows that the function $S(t)$ is similar to an elliptic integral.

3. Pressure-driven flow through an annular hole in a wall

The pressure-driven flow through a circular hole in a wall was first described by Sampson (1891) and is conveniently given by Happel & Brenner (1973) in terms of oblate spheroidal coordinates (λ,ζ) defined by

$$z = \lambda\zeta, \quad \rho = (1=\lambda^2)^{\frac{1}{2}}(1-\zeta^2)^{\frac{1}{2}} \tag{3.1}$$

with $-\infty < \lambda < \infty$, $0 \leqslant \zeta \leqslant 1$. Alternatively, the appropriate function ϕ in the representation (2.4) is given (Davis 1991*b*) by

$$\phi^{(s)} = \frac{\Delta P}{2\pi\mu}\zeta[1+\lambda\tan^{-1}\lambda], \tag{3.2}$$

where ΔP is the pressure drop that drives the flow. In particular the velocity in the hole is

$$v_z^{(s)}(\rho,0) = (\phi^{(s)})_{\lambda=0} = \frac{\Delta P}{2\pi\mu}(1-\rho^2)^{\frac{1}{2}} \quad (0 \leqslant \rho < 1) \tag{3.3}$$

and the flux of fluid is $\Delta P/3\mu$.

Consider now the extrusion of viscous fluid through an annular hole at $z = 0$, $a < \rho < 1$. If the total velocity field is denoted by $\boldsymbol{v}^{(s)} - \boldsymbol{v}$ and the representation (2.4) is used for the negative perturbation to the Sampson flow due to the presence of a disk of radius a, then the modified forms of (2.7), (2.8) are

$$v_z(\rho, 0) = \frac{\Delta P}{2\pi\mu}\int_0^\infty A(k)\, J_0(k\rho)\,\mathrm{d}k, \tag{3.4}$$

$$[p]_{z=0} = \frac{2\Delta P}{\pi}\int_0^\infty kA(k)\, J_0(k\rho)\,\mathrm{d}k, \tag{3.5}$$

where $A(k)$ is now to be determined by the boundary conditions $v_z(\rho,0) = v_z^{(s)}(\rho,0)$ for $0 \leqslant \rho \leqslant a$ and $\rho \geqslant 1$ and $[p]_{z=0} = 0$ for $a < \rho < 1$. Evidently a set of triple integral equations is obtained but again a simpler procedure than the standard approach is available for their solution, this time by using representations of the unknown pressure jumps at the disk and plane. Thus, following Davis (1990), write

$$[p]_{z=0} = -\frac{2\Delta P}{\pi\rho}\left[H(a-\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{tS(t)}{(t^2-\rho^2)^{\frac{1}{2}}}\mathrm{d}t + H(\rho-1)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_1^\rho \frac{tX(t)}{(\rho^2-t^2)^{\frac{1}{2}}}\mathrm{d}t\right], \tag{3.6}$$

whence the inversion of (3.5) yields

$$A(k) = \int_0^a S(t)\cos kt\,\mathrm{d}t - \int_1^\infty X(t)\sin kt\,\mathrm{d}t. \tag{3.7}$$

Meanwhile, application of the u-derivatives of the identities (2.13) to the velocity conditions

$$\int_0^\infty A(k)\, J_0(k\rho)\,\mathrm{d}k = \begin{cases} 0 & (\rho \geqslant 1) \\ (1-\rho^2)^{\frac{1}{2}} & (0 \leqslant \rho \leqslant a), \end{cases}$$

obtained by substitution of (3.3) and (3.4), shows that

$$\int_0^\infty A(k)\sin ku\,\mathrm{d}k = 0 \quad (u \geqslant 1),$$

$$\int_0^\infty A(k)\cos ku\,\mathrm{d}k = 1 - \frac{u}{2}\ln\left(\frac{1+u}{1-u}\right) \quad (0 \leqslant u \leqslant a).$$

Then, on substitution of (3.7), these equations become

$$u\int_0^a \frac{S(t)}{u^2-t^2}\mathrm{d}t = \frac{\pi}{2}X(u) \quad (u \geqslant 1), \tag{3.8}$$

$$\frac{\pi}{2}S(u) - \int_1^\infty \frac{tX(t)}{t^2-u^2}\mathrm{d}t = 1 - \frac{u}{2}\ln\left(\frac{1+u}{1-u}\right) \quad (0 \leqslant u \leqslant a),$$

and hence $S(t)$ satisfies the integral equation

$$S(u) - \frac{2}{\pi^2}\int_0^a \left[u\ln\left(\frac{1+u}{1-u}\right) - v\ln\left(\frac{1+v}{1-v}\right)\right]\frac{S(v)}{u^2-v^2}\mathrm{d}v = \frac{2}{\pi} - \frac{u}{\pi}\ln\left(\frac{1+u}{1-u}\right) \quad (0 \leqslant u \leqslant a). \tag{3.9}$$

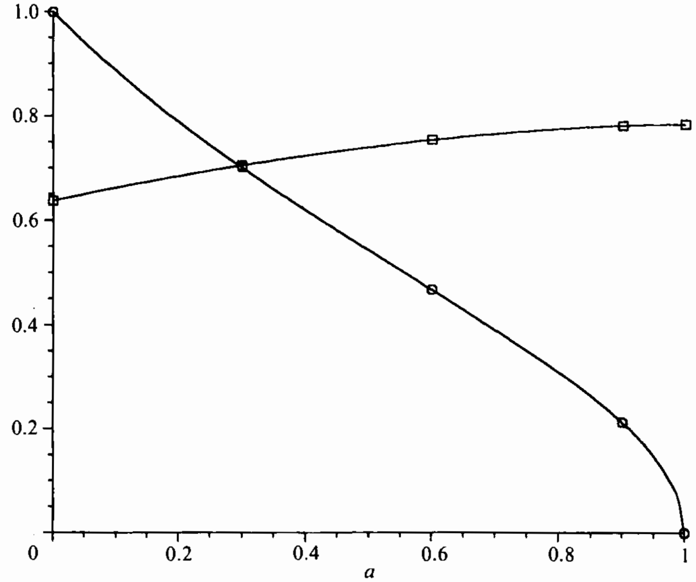

FIGURE 1. The dimensionless quantities $M(1-a^2)^{-\frac{3}{2}}$ (○) and C (□) that measure the flux of fluid $M\Delta P/3\mu$ and the force $4\Delta PaC$ on the central fixed disk in the pressure-driven extrusion of viscous fluid through an annular hole of radii a, 1 in a wall. This scaling of M facilitates its graphical display.

Since $-p$ is the additional pressure due to the replacement of the circular hole by an annular hole in the wall, the force $4\Delta PaC$ exerted by the fluid on the disk is given, from (3.6), by

$$C = \frac{1}{a}\int_0^a S(t)\,\mathrm{d}t. \tag{3.10}$$

Evidently, the functions $S(t)$ $(0 \leqslant t < a)$ and $X(t)$ $(t > 1)$ may be identified, as in Davis (1990), as density functions for axisymmetric distributions of point force singularities on the rigid plane.

The total normal velocity in the annular hole is given from (3.3), (3.4) and (3.7) by

$$\begin{aligned} v_z^{(s)}(\rho,0) - v_z(\rho,0) &= \frac{\Delta P}{2\pi\mu}\left[(1-\rho^2)^{\frac{1}{2}} - \int_0^a \frac{S(t)}{(\rho^2-t^2)^{\frac{1}{2}}}\,\mathrm{d}t + \int_1^\infty \frac{X(t)}{(t^2-\rho^2)^{\frac{1}{2}}}\,\mathrm{d}t\right] \\ &= \frac{\Delta P}{2\pi\mu}\left[(1-\rho^2)^{\frac{1}{2}} - \frac{2}{\pi}\int_0^a \frac{S(t)}{(\rho^2-t^2)^{\frac{1}{2}}}\sin^{-1}\left[\left(\frac{1-\rho^2}{1-t^2}\right)^{\frac{1}{2}}\right]\mathrm{d}t\right] \quad (a \leqslant \rho \leqslant 1) \end{aligned} \tag{3.11}$$

after substitution of (3.8). Hence the flux $M\Delta P/3\mu$ of fluid through the annular hole is determined by

$$M = (1-a^2)^{\frac{3}{2}} - \frac{6}{\pi}\int_0^a S(t)\left\{(1-a^2)^{\frac{1}{2}} - (a^2-t^2)^{\frac{1}{2}}\tan^{-1}\left[\left(\frac{1-a^2}{a^2-t^2}\right)^{\frac{1}{2}}\right]\right\}\mathrm{d}t. \tag{3.12}$$

Values of the dimensionless drag force C and flux factor M can be computed by applying the method described in the Appendix to the integral equation (3.9). The variations of C and $M(1-a^2)^{-\frac{3}{2}}$ with a are displayed in figure 1 while velocity profiles in the hole, significantly different from those for flow between concentric cylinders,

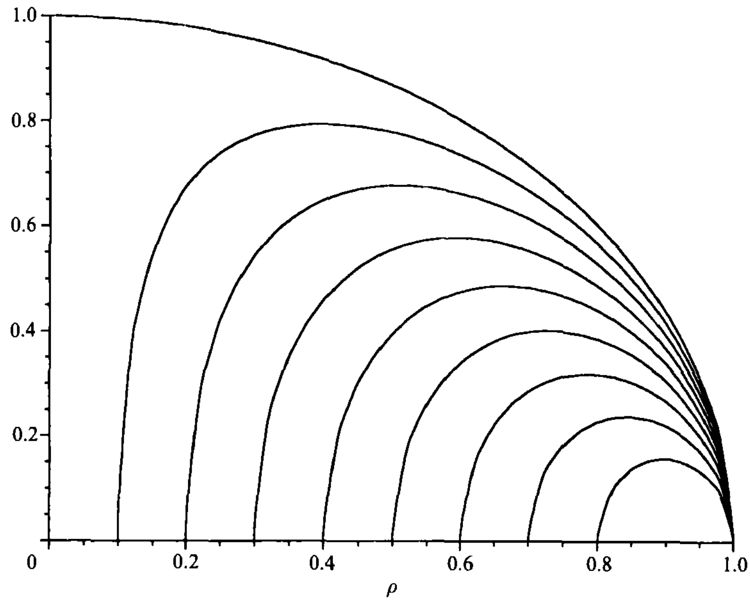

FIGURE 2. Velocity profiles in the hole for $a = 0$ to 0.8 in increments of 0.1.

are shown in figure 2 at increments of 0.1 in a. As in table 1, $N = 5$ is sufficient for convergence of the computations.

It is of interest to consider the extreme values of a. For $a \ll 1$, the integral equation (3.9) yields

$$S(t) \sim \frac{2}{\pi}\left(1+\frac{4a}{\pi^2}\right) \sim C \tag{3.13}$$

in which the leading term can be anticipated by observing that the small disk essentially sees a uniform stream with speed $v_z^{(s)}(0,0) = \Delta P/2\pi\mu$. Substitution of this linear approximation for $S(t)$ in (3.12) then yields the quadratic estimate

$$M \sim 1-\frac{12a}{\pi^2}-\frac{48}{\pi^4}a^2$$

in which the linear term can be deduced from a simple application of the reciprocal theorem (A. Acrivos 1990, private communication). Evidently $M(a)$ has a point of inflexion; the preferred quantity $M(1-a^2)^{-\frac{3}{2}}$ corresponds to M/a^3 for the disk and measures, according to (3.12), the fraction of the Sampson flux that passes through the annulus. Also, substitution of (3.13) in (3.8) and (3.11) yields

$$X(u) \sim \frac{1}{\pi}C\ln\left(\frac{u+a}{u-a}\right) \sim \frac{2a}{\pi u}C$$

and hence the velocity profile

$$v_z^{(s)}(\rho,0)-v_z(\rho,0) = \frac{\Delta P}{2\pi\mu}\left[(1-\rho^2)^{\frac{1}{2}}-\frac{2}{\pi}\left(1+\frac{4a}{\pi^2}\right)\left\{\sin^{-1}\left(\frac{a}{\rho}\right)-\frac{2a}{\pi}\sin^{-1}\rho\right\}\right]+O(a^2). \tag{3.14}$$

Thus, as suggested by figure 2, the deviation from the Sampson profile is $O(a)$ except when $\rho = 0(a)$, i.e.

$$\lim_{a\to 0} v_z(\rho,0) = 0 \quad \text{for any} \quad \rho > 0.$$

The boundary-layer behaviour causes the profile to steepen as $a \to 0$, a phenomenon that tends to confirm the indicated structure of the flow through an annular disk when $(1-a)$ is small.

For $1-a \ll 1$, an approximate solution of (3.9) is unavailable but evidently $C \to \frac{1}{4}\pi$ as $a \to 1$ because the force exerted on the disk must approach $\pi a^2 \Delta P$. For an estimate of M in this limit, A. Acrivos (1990, private communication) has used the method of Hasimoto (1958) to obtain

$$M \sim \frac{3\pi^2}{16}(1-a)^2[1+O(1-a)]. \tag{3.15}$$

By exploiting the similarity between the equations governing creeping flow through holes or slots and those for potential flow past the complementary disks or strips, Hasimoto showed that the flux per unit pressure drop for a slit of width $2h$ is $\pi h^2/8\mu$ per unit length. On regarding the narrow annular hole as a slit of width $(1-a)$ and length 2π, the estimate (3.15) is readily deduced. The computed values $M(0.98) = 7.239 \times 10^{-4}$, $M(0.99) = 1.775 \times 10^{-4}$ indicate good agreement.

4. Disk moving sideways between parallel walls

A thin rigid disk of unit radius translates steadily in its own plane with speed U in incompressible viscous fluid that is at rest at infinity and possibly confined between parallel walls at distance H from the plane of the disk. Cylindrical polar coordinates (ρ, θ, z) are chosen so that the disk is instantaneously at $z=0$ $(0 \leqslant \rho \leqslant 1,\ -\pi < \theta \leqslant \pi)$ and moving in the $\theta = 0$ direction. According to Ranger (1978), the creeping-flow equations (2.1), (2.2) can be satisfied by writing

$$\boldsymbol{v} = (\nabla \times)^2\left[\frac{\psi}{\rho}\hat{\boldsymbol{z}}\cos\theta\right] + \nabla \times \left[\frac{\chi}{\rho}\hat{\boldsymbol{z}}\sin\theta\right] \tag{4.1}$$

where the functions $\psi(\rho, z)$, $\chi(\rho, z)$ satisfy

$$L_{-1}^2 \psi = 0 = L_{-1}\chi. \tag{4.2}$$

The components of $\boldsymbol{v}$ are given by

$$\left.\begin{aligned} v_z &= -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right)\cos\theta, \\ v_\rho &= \left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial z}\right) + \frac{\chi}{\rho^2}\right]\cos\theta, \\ v_\theta &= -\left[\frac{1}{\rho^2}\frac{\partial\psi}{\partial z} + \frac{\partial}{\partial\rho}\left(\frac{\chi}{\rho}\right)\right]\sin\theta, \end{aligned}\right\} \tag{4.3}$$

but it is more convenient to replace v_ρ, v_θ by their Cartesian counterparts

$$\left.\begin{aligned} v_x &= \tfrac{1}{2}\rho\frac{\partial}{\partial\rho}\left[\rho^{-2}\left(\frac{\partial\psi}{\partial z} - \chi\right)\right]\cos 2\theta + \frac{1}{2\rho}\frac{\partial}{\partial\rho}\left(\frac{\partial\psi}{\partial z} + \chi\right), \\ v_y &= \tfrac{1}{2}\rho\frac{\partial}{\partial\rho}\left[\rho^{-2}\left(\frac{\partial\psi}{\partial z} - \chi\right)\right]\sin 2\theta. \end{aligned}\right\} \tag{4.4}$$

The boundary conditions

$$v_z = 0, \quad v_x = U, \quad v_y = 0 \quad \text{at} \quad z = 0 \quad (0 \leqslant \rho \leqslant 1) \tag{4.5}$$

are satisfied by choosing ψ to be an odd function of z and requiring that

$$\frac{\partial \psi}{\partial z}+\chi = \rho^2 U, \quad \frac{\partial \psi}{\partial z}-\chi = -\tfrac{1}{3}\alpha\rho^2 U \quad \text{on} \quad z=0 \quad (0 \leqslant \rho \leqslant 1), \tag{4.6}$$

where α is a constant to be determined.

In the absence of the parallel walls, i.e. for infinite fluid, an appropriate form for ψ is

$$\psi = z\Psi, \quad L_{-1}\Psi = 0$$

and then Ψ and χ have similar forms, namely

$$\begin{bmatrix}\psi \\ \chi\end{bmatrix} = \frac{2U}{\pi}\int_0^\infty \rho J_1(k\rho)\begin{bmatrix}A(k)\\ B(k)\end{bmatrix} e^{-k|z|}\, dk. \tag{4.7}$$

Hence, from (4.6)

$$\frac{2}{\pi}\int_0^\infty J_1(k\rho)\begin{bmatrix}A+B\\ A-B\end{bmatrix} dk = \begin{bmatrix}1\\ -\tfrac{1}{3}\alpha\end{bmatrix}\rho \quad (0 \leqslant \rho \leqslant 1). \tag{4.8}$$

The tangential stress discontinuities at $z=0$ are given, after substitution of (4.7) in (4.4), by

$$\left.\begin{aligned}\mu\left[\frac{\partial v_x}{\partial z}\right]_{0-}^{0+} &= -\frac{2U\mu}{\pi}\rho\frac{d}{d\rho}\left\{\frac{1}{\rho}\int_0^\infty kJ_1(k\rho)(2A-B)\, dk\right\}\cos 2\theta \\ &\quad -\frac{2U}{\pi\rho}\frac{d}{d\rho}\left\{\rho\int_0^\infty kJ_1(k\rho)(2A+B)\, dk\right\}, \\ \mu\left[\frac{\partial v_y}{\partial z}\right]_{0-}^{0+} &= -\frac{2U\mu}{\pi}\rho\frac{d}{d\rho}\left\{\frac{1}{\rho}\int_0^\infty kJ_1(k\rho)(2A-B)\, dk\right\}\sin 2\theta.\end{aligned}\right\} \tag{4.9}$$

Hence continuity of stresses at $z=0$, $\rho>1$ is achieved by imposing the conditions

$$\int_0^\infty kJ_1(k\rho)\begin{bmatrix}2A+B\\ 2A-B\end{bmatrix} dk = \begin{bmatrix}1\\ 0\end{bmatrix}\frac{8\beta}{3\rho} \quad (\rho > 1), \tag{4.10}$$

where β is another undetermined constant.

Note that it is essential here, because different combinations of $A(k)$ and $B(k)$ occur in the velocity and stress conditions, to refrain from evaluating the ρ-derivatives in (4.4) and (4.9) in order to have Bessel functions of common order in (4.8) and (4.10). The solution is to be made unique by choosing the constants α and β so that the velocities are bounded at the rim $\rho = 1$. β may be interpreted as a dimensionless drag coefficient by evaluating the total tangential force exerted by the fluid on the disk against its motion, namely

$$-\mu\int_{-\pi}^{\pi}\int_0^1\left[\frac{\partial v_x}{\partial z}\right]_{0-}^{0+}\rho\, d\rho\, d\theta = 4U\mu\int_0^\infty kJ_1(k)(2A+B)\, dk = \tfrac{32}{3}U\mu\beta \tag{4.11}$$

by substitution of (4.9) and the condition (4.10).

After rearranging (4.10) in the form

$$\int_0^\infty kJ_1(k\rho)\begin{bmatrix}A+B\\ A-B\end{bmatrix} dk = \begin{bmatrix}1\\ -\tfrac{1}{3}\end{bmatrix}\frac{2\beta}{\rho} \quad (\rho > 1) \tag{4.12}$$

it is seen that (4.8) and (4.12) furnish two sets of dual integral equations to which the formulae of Sneddon (1966, §4.3) are applicable in a manner similar to that described by Davis (1991*b*) for the shear flow disturbances due to a hole in the bounding wall. The constants α and β are chosen to eliminate terms of order $(\rho^2-1)^{-\frac{1}{2}}$ in the velocity components in the plane of the disk. Here it suffices to observe that the identities (2.11) and

$$\int_0^\infty J_1(k\rho)\frac{\sin k}{k}\mathrm{d}k = \frac{1}{\rho}[1-H(1-\rho)(1-\rho^2)^{\frac{1}{2}}] \tag{4.13}$$

imply that (4.12) and a derivative of (4.8), namely

$$\int_0^\infty kJ_0(k\rho)\begin{bmatrix}A+B\\A-B\end{bmatrix}\mathrm{d}k = \pi\begin{bmatrix}1\\-\frac{1}{3}\alpha\end{bmatrix} \quad (0 \leqslant \rho \leqslant 1),$$

can be satisfied by setting $\alpha = \beta = 1$ with

$$B = 2A = \frac{4\sin k}{3k^2}. \tag{4.14}$$

Thus, on substitution in (4.7) and (4.11), the drag force is $32U\mu/3$ and

$$\tfrac{1}{2}\chi = \Psi = z^{-1}\psi = \frac{4U}{3\pi}\int_0^\infty \rho J_1(k\rho)\frac{\sin k}{k^2}\mathrm{e}^{-k|z|}\mathrm{d}k,$$

whose evaluation in closed form is unavailable. However, the corresponding velocity components determined by (4.3) have relatively simple evaluations in terms of the oblate spheroidal coordinates defined by (3.1) with $\lambda > 0$, $-1 \leqslant \zeta \leqslant 1$ for the region external to the disk. Thus, with the pressure given by

$$p = \frac{\mu}{\rho}\frac{\partial}{\partial z}(L_{-1}\psi)\cos\theta,$$

it follows by various manipulations of the identity

$$\int_0^\infty \mathrm{e}^{-k\lambda}J_\nu(k\rho)\,\mathrm{d}k = \frac{[(\lambda^2+\rho^2)^{\frac{1}{2}}-\lambda]^\nu}{\rho^\nu(\lambda^2+\rho^2)^{\frac{1}{2}}} \quad (\mathrm{Re}\,\lambda > 0) \tag{4.15}$$

(Gradshteyn & Ryzhik 1980, §6.611) that

$$\left.\begin{aligned}(v_x, v_y) &= (\cos 2\theta, \sin 2\theta)\frac{2U}{3\pi}\frac{\lambda^3(1-\zeta^2)}{(\lambda^2+1)(\lambda^2+\zeta^2)}\\ &\quad +(1,0)\frac{2U}{3\pi}\left[3\tan^{-1}\left(\frac{1}{\lambda}\right)-\frac{\lambda\zeta^2}{\lambda^2+\zeta^2}\right],\\ (v_z, p) &= (\lambda\zeta, 2\mu)\frac{4U}{3\pi}\cos\theta\frac{\lambda}{\lambda^2+\zeta^2}\left(\frac{1-\zeta^2}{\lambda^2+1}\right)^{\frac{1}{2}}.\end{aligned}\right\} \tag{4.16}$$

In particular, the tangential stress discontinuities on the disk are

$$\mu\left[\frac{\partial v_x}{\partial z}\right]_{0-}^{0+} = -\frac{16U\mu}{3\pi}(1-\rho^2)^{-\frac{1}{2}}, \quad \mu\left[\frac{\partial v_y}{\partial z}\right] = 0 \quad (0 \leqslant \rho \leqslant 1),$$

in which the choice of the constants α and β has eliminated terms of order $(1-\rho^2)^{-\frac{3}{2}}$. This was not achieved by Ranger (1978) who omitted the second Fourier component from (4.4) in a solution based on complementary integral representations.

Now consider how the above analysis should be modified to take account of rigid walls at $z = \pm H$. The total velocity field is then determined by $(\psi - \psi_B)$ and $(\chi - \chi_B)$ which satisfy (4.2) with ψ, χ constructed as in (4.7) and ψ_B, χ_B determined by the no-slip conditions at $z = \pm H$. Thus

$$\left.\begin{aligned} \chi_B &= \frac{2U}{\pi}\int_0^\infty \rho J_1(k\rho)\, C(k)\cosh kz\,\mathrm{d}k, \\ \psi_B &= \frac{2U}{\pi}\int_0^\infty \rho J_1(k\rho)\,[D(k)\sinh kz + E(k)\, z\cosh kz]\,\mathrm{d}k, \end{aligned}\right\} \tag{4.17}$$

where the unknown functions are chosen so that

$$\chi = \chi_B, \quad \psi = \psi_B, \quad \frac{\partial\psi}{\partial z} = \frac{\partial\psi_B}{\partial z} \quad \text{at} \quad z = H.$$

Thus, by reference to (4.7),

$$C\cosh kH = B\,\mathrm{e}^{-kH},$$

$$D\sinh kH + EH\cosh kH = AH\,\mathrm{e}^{-kH},$$

$$kD\cosh kH + E(\cosh kH + kH\sinh kH) = A(1-kH)\,\mathrm{e}^{-kH}$$

and hence, from (4.17),

$$\left.\begin{aligned} (\chi_B)_{z=0} &= \frac{2U}{\pi}\int_0^\infty \rho J_1(k\rho)\, B(k)\, \Delta(k)\,\mathrm{d}k, \\ \left(\frac{\partial\psi_B}{\partial z}\right)_{z=0} &= \frac{2U}{\pi}\int_0^\infty \rho J_1(k\rho)\, A(k)\, \Gamma(k)\,\mathrm{d}k, \end{aligned}\right\} \tag{4.18}$$

where

$$\Delta(k) = \frac{\mathrm{e}^{-kH}}{\cosh kH}, \quad \Gamma(k) = \frac{1 - 2kH + 2k^2H^2 - \mathrm{e}^{-2kH}}{\sinh 2kH - 2kH}. \tag{4.19}$$

The reflected velocity field must be included in the boundary conditions (4.5) but does not contain any stress discontinuities. Hence, (4.8) is replaced by

$$\frac{2U}{\pi}\int_0^\infty \rho J_1(k\rho)\begin{bmatrix} A+B \\ A-B \end{bmatrix}\mathrm{d}k = \begin{bmatrix} 1 \\ -\frac{1}{3}\alpha \end{bmatrix}\rho^2 U + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \partial\psi_B/\partial z \\ \chi_B \end{bmatrix}_{z=0} \quad (0 \leqslant \rho \leqslant 1) \tag{4.20}$$

but (4.10) is retained.

Guided by the identity (4.13) and the solution (4.14) for the disk in isolation, write

$$\int_0^\infty kJ_1(k\rho)\begin{bmatrix} 2A+B \\ 2A-B \end{bmatrix}\mathrm{d}k = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{8\beta}{3\rho}[1-(1-\rho^2)^{\frac{3}{2}}] - \frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^1 \begin{bmatrix} W(t) \\ X(t) \end{bmatrix}\frac{\mathrm{d}t}{(t^2-\rho^2)^{\frac{1}{2}}} \quad (0 \leqslant \rho \leqslant 1). \tag{4.21}$$

Then the inversion of the Hankel transforms defined by (4.10) and (4.21) yields

$$\left.\begin{aligned} 2A+B &= \frac{8\beta}{3}\frac{\sin k}{k^2} + \int_0^1 W(t)\sin kt\,\mathrm{d}t, \\ 2A-B &= \int_0^1 X(t)\sin kt\,\mathrm{d}t, \end{aligned}\right\} \tag{4.22}$$

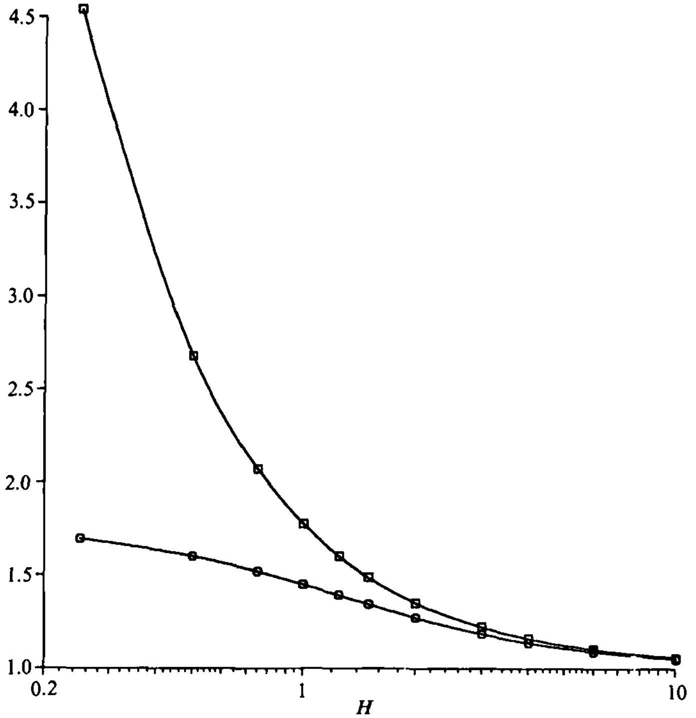

FIGURE 3. The dimensionless drag coefficient β (□) and the constant α (○) for a unit disk moving sideways at distance H from each of two parallel walls.

and hence, for $0 < \rho < 1$,

$$\int_0^\infty \rho J_1(k\rho)\begin{bmatrix}2A+B\\2A-B\end{bmatrix}\mathrm{d}k = \begin{bmatrix}1\\0\end{bmatrix}\frac{2\pi\beta}{3}\rho^2 + \int_0^\rho \begin{bmatrix}W(t)\\X(t)\end{bmatrix}\frac{t\,\mathrm{d}t}{(\rho^2-t^2)^{\frac{1}{2}}}.$$

But the left-hand side is determined by (4.20) and thus

$$\int_0^\rho \begin{bmatrix}W(t)\\X(t)\end{bmatrix}\frac{t\,\mathrm{d}t}{(\rho^2-t^2)^{\frac{1}{2}}} = \frac{\pi\rho^2}{4}\begin{bmatrix}3-\frac{1}{3}(\alpha+8\beta)\\1-\alpha\end{bmatrix}$$
$$+\begin{bmatrix}2 & 1\\2 & -1\end{bmatrix}\int_0^\infty \rho J_1(k\rho)\begin{bmatrix}A(k)\,\Gamma(k)\\B(k)\,\Delta(k)\end{bmatrix}\mathrm{d}k \quad (0 \leqslant \rho \leqslant 1),$$

after substitution of (4.18). Abel inversion then shows that

$$\begin{bmatrix}W(t)\\X(t)\end{bmatrix} = t\begin{bmatrix}3-\frac{1}{3}(\alpha+8\beta)\\1-\alpha\end{bmatrix} + \frac{2}{\pi}\int_0^\infty \sin kt\begin{bmatrix}2 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}A(k)\,\Gamma(k)\\B(k)\,\Delta(k)\end{bmatrix}\mathrm{d}k \quad (0 \leqslant t \leqslant 1),$$

and finally substitution for A, B from (4.22) yields the following coupled integral equations for $W(t)$, $X(t)$:

$$\begin{bmatrix}W(t)\\X(t)\end{bmatrix} - \frac{1}{\pi}\int_0^1\int_0^\infty \begin{bmatrix}(\Gamma+\Delta) & (\Gamma-\Delta)\\(\Gamma-\Delta) & (\Gamma+\Delta)\end{bmatrix}\sin kt\sin ku\,\mathrm{d}k\begin{bmatrix}W(u)\\X(u)\end{bmatrix}\mathrm{d}u$$
$$= t\begin{bmatrix}3-\frac{1}{3}(\alpha+8\beta)\\1-\alpha\end{bmatrix} + \frac{8\beta}{3\pi}\int_0^\infty \begin{bmatrix}\Gamma+\Delta\\\Gamma-\Delta\end{bmatrix}\frac{\sin kt\sin k}{k^2}\mathrm{d}k \quad (0 \leqslant t \leqslant 1), \quad (4.23)$$

where the matrix of functions in the kernels is defined by (4.19) and can be simplified to the form

$$\begin{bmatrix} \Gamma+\Delta & \Gamma-\Delta \\ \Gamma-\Delta & \Gamma+\Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{2\,\mathrm{e}^{-kH}}{\cosh kH} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \left(\frac{kH - \tanh kH}{\dfrac{\sinh 2kH}{2kH} - 1} \right).$$

Inspection of (4.21) shows that the rim singularities in the pressure are minimized by imposing the conditions

$$W(1) = 0 = X(1)$$

and these ensure a unique solution of (4.23) by determining the dimensionless drag coefficient β and the other constant α.

Figure 3 displays computed values of β and α obtained as described in the Appendix with $N = 5$ and $N = 3$ for convergence in the ranges $0.25 \leqslant H \leqslant 2$ and $H > 2$ respectively. For $H \gg 2$, the disk is small compared to its distance from each wall and hence the solution is determined, to leading order, by a moving point-force singularity. Thus the frictional variation in drag, due to the walls, must to order H^{-1} agree with that calculated by Ganatos, Pfeffer & Weinbaum (1980) for the sphere and Liran & Mochan (1976) for the stokeslet translating between parallel walls.

5. Annular disk moving sideways

Suppose that a thin rigid annular disk of radii $a, 1$ $(a < 1)$ translates in infinite fluid with the same motion as the disk considered in the previous section. Equations (4.1)–(4.4) remain valid but the boundary conditions (4.5) are now confined to the interval $a \leqslant \rho \leqslant 1$ and the corresponding modification of conditions (4.6), namely

$$\frac{\partial \psi}{\partial z} + \chi = \left(\rho^2 + \frac{2}{\pi}\gamma\right)U, \quad \frac{\partial \psi}{\partial z} - \chi = -\tfrac{1}{3}\alpha\rho^2 U \quad \text{on} \quad z = 0 \quad (a \leqslant \rho \leqslant 1)$$

involves two undetermined constants, α and γ. The solution form (4.7) for ψ/z and χ then implies that (4.8) must be replaced by

$$\int_0^\infty J_1(k\rho) \begin{bmatrix} A+B \\ A-B \end{bmatrix} \mathrm{d}k = \begin{bmatrix} 1 \\ -\frac{1}{3}\alpha \end{bmatrix} \frac{\pi}{2}\rho + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{\gamma}{\rho} \quad (a \leqslant \rho \leqslant 1) \tag{5.1}$$

and also, from (4.9), that continuity of stresses at $z = 0$, $\rho > 1$ and $z = 0$, $0 \leqslant \rho < a$ is achieved by imposing the conditions (4.10) and

$$\int_0^\infty kJ_1(k\rho) \begin{bmatrix} 2A+B \\ 2A-B \end{bmatrix} \mathrm{d}k = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi\delta\rho \quad (0 \leqslant \rho < a). \tag{5.2}$$

Thus there are four constants α, β, γ, δ to be determined by requiring both velocity components to remain bounded as either rim of the annular disk is approached.

After rearranging (4.10) and (5.2) in the forms

$$\left.\begin{aligned} \int_0^\infty kJ_1(k\rho) \begin{bmatrix} A+B \\ A-B \end{bmatrix} \mathrm{d}k &= \begin{bmatrix} -1 \\ 3 \end{bmatrix} \frac{\pi}{4}\delta\rho \quad (0 \leqslant \rho < a), \\ \int_0^\infty kJ_1(k\rho) \begin{bmatrix} A+B \\ A-B \end{bmatrix} \mathrm{d}k &= \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix} \frac{2\beta}{\rho} \quad (\rho > 1), \end{aligned}\right\} \tag{5.3}$$

these equations, together with (5.1), constitute two sets of triple integral equations for the unknown functions $k(A+B)$ and $k(A-B)$. However, for the simplest construction of a solution, it is advantageous to proceed otherwise. First α is eliminated by considering, as in the previous section, a derivative of (5.1), namely

$$\int_0^\infty kJ_0(k\rho)\begin{bmatrix}A+B\\A-B\end{bmatrix}\mathrm{d}k = \pi\begin{bmatrix}1\\-\frac{1}{3}\alpha\end{bmatrix} \quad (a \leqslant \rho \leqslant 1).$$

But, by analogy with (2.9) and (2.12), this can be satisfied by writing

$$k\begin{bmatrix}A+B\\A-B\end{bmatrix} = 2\begin{bmatrix}1\\-\frac{1}{3}\alpha\end{bmatrix}\frac{\sin k}{k} - \int_0^a t^{-1}\begin{bmatrix}F_+(t)\\F_-(t)\end{bmatrix}\sin kt\,\mathrm{d}t - \int_1^\infty\begin{bmatrix}G_+(t)\\G_-(t)\end{bmatrix}\cos kt\,\mathrm{d}t. \tag{5.4}$$

So, on substitution in (4.7), the velocity components v_x, v_y in (4.4) are determined at $z = 0$ by

$$\begin{aligned}\frac{1}{2\rho}\frac{\partial}{\partial\rho}[\Psi+\chi]_{z=0} &= \frac{U}{\pi}\int_0^\infty k(A+B)\,J_0(k\rho)\,\mathrm{d}k\\ &= UH(1-\rho) - \frac{U}{\pi}H(a-\rho)\int_\rho^a \frac{t^{-1}F_+(t)}{(t^2-\rho^2)^{\frac{1}{2}}}\mathrm{d}t - \frac{U}{\pi}H(\rho-1)\int_1^\rho \frac{G_+(t)}{(\rho^2-t^2)^{\frac{1}{2}}}\mathrm{d}t,\end{aligned} \tag{5.5a}$$

$$\begin{aligned}\tfrac{1}{2}\rho\frac{\partial}{\partial\rho}[\rho^{-2}(\Psi-\chi)]_{z=0} &= -\frac{U}{\pi}\int_0^\infty k(A-B)\,J_2(k\rho)\,\mathrm{d}k\\ &= \frac{2\alpha U}{3\pi\rho}(\rho^2-1)^{\frac{1}{2}}H(\rho-1) + \frac{2U}{\pi\rho^2}\int_0^a F_-(t)\,\mathrm{d}t\\ &\quad - \frac{UH(a-\rho)}{\pi\rho^2}\int_\rho^a F_-(t)\left[\frac{t}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{(t^2-\rho^2)^{\frac{1}{2}}}{t}\right]\mathrm{d}t\\ &\quad + \frac{UH(\rho-1)}{\pi\rho^2}\int_1^\rho G_-(t)\left[(\rho^2-t^2)^{\frac{1}{2}} - \frac{t^2}{(\rho^2-t^2)^{\frac{1}{2}}}\right]\mathrm{d}t\end{aligned} \tag{5.5b}$$

after suitable manipulations of the identity (4.15). Thus the use of the representation (5.4) in terms of continuous, integrable functions $t^{-1}F_\pm(t)$ $(0 \leqslant t \leqslant a)$ and $G_\pm(t)$ $(t > 1)$ ensures that the velocity components exhibit the required square-root behaviour at each rim and it remains to identify three consistency conditions that will determine α, β and δ. Because of the switch between J_0 and J_2, the second velocity expression (5.5b) contains a term that must be eliminated in order to satisfy the disk conditions (4.5). Thus

$$\int_0^a F_-(t)\,\mathrm{d}t = 0. \tag{5.6}$$

Also, since the limit $\rho \to \infty$ in (5.3) implies that

$$\lim_{k\to 0} k\begin{bmatrix}A+B\\A-B\end{bmatrix} = 2\beta\begin{bmatrix}1\\-\frac{1}{3}\end{bmatrix},$$

it follows by letting $k \to 0$ in (5.4) that

$$\int_1^\infty\begin{bmatrix}G_+(t)\\G_-(t)\end{bmatrix}\mathrm{d}t = 2\begin{bmatrix}1-\beta\\\frac{1}{3}(\beta-\alpha)\end{bmatrix}. \tag{5.7}$$

The stress conditions (5.3) now determine the unknown functions. Direct substitution of (5.4) yields two pairs of Abel integral equations which can be inverted by standard methods. However the algebra is much abbreviated, as in §2, by applying the identities

$$\frac{\partial}{\partial u} u \int_0^u \frac{J_1(k\rho)}{(u^2-\rho^2)^{\frac{1}{2}}} \mathrm{d}\rho = \frac{\partial}{\partial u}\left(\frac{1-\cos ku}{k}\right) = \sin ku,$$

$$\frac{\partial}{\partial u} u \int_u^\infty \frac{J_1(k\rho)}{(\rho^2-u^2)} \mathrm{d}\rho = \frac{\partial}{\partial u}\left(\frac{\sin ku}{k}\right) = \cos ku$$

to the respective stress conditions (5.3) to obtain

$$\int_0^\infty k \begin{bmatrix} A+B \\ A-B \end{bmatrix} \sin ku \,\mathrm{d}k = \frac{\pi}{2}\delta u \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad (0 \leqslant u \leqslant a),$$

$$\int_0^\infty k \begin{bmatrix} A+B \\ A-B \end{bmatrix} \cos ku \,\mathrm{d}k = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (u \leqslant 1).$$

Except for the inhomogeneous term, the substitution of (5.4) involves the same calculation as for (2.14) and (2.15) and hence

$$\begin{bmatrix} 1 \\ -\frac{1}{3}\alpha \end{bmatrix} \ln\left(\frac{1+u}{1-u}\right) - \frac{\pi}{2u}\begin{bmatrix} F_+(u) \\ F_-(u) \end{bmatrix} + u\int_1^\infty \begin{bmatrix} G_+(t) \\ G_-(t) \end{bmatrix} \frac{\mathrm{d}t}{t^2-u^2} = \frac{\pi}{2}\delta u \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad (0 \leqslant u \leqslant a),$$

$$\int_0^a \begin{bmatrix} F_+(t) \\ F_-(t) \end{bmatrix} \frac{\mathrm{d}t}{u^2-t^2} = \frac{\pi}{2}\begin{bmatrix} G_+(u) \\ G_-(u) \end{bmatrix} \quad (u \geqslant 1),$$

i.e. $F_+(u), F_-(u)$ satisfy the disjoint integral equations

$$\begin{bmatrix} F_+(u) \\ F_-(u) \end{bmatrix} - \frac{2u^2}{\pi^2}\int_0^a \left[\frac{1}{u}\ln\left(\frac{1+u}{1-u}\right) - \frac{1}{v}\ln\left(\frac{1+v}{1-v}\right)\right]\begin{bmatrix} F_+(v) \\ F_-(v) \end{bmatrix}\frac{\mathrm{d}v}{u^2-v^2}$$

$$= \frac{2u}{\pi}\begin{bmatrix} 1 \\ -\frac{1}{3}\alpha \end{bmatrix} \ln\left(\frac{1+u}{1-u}\right) + \delta u^2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad (0 \leqslant u \leqslant a), \tag{5.8}$$

with the ratio $\delta : \alpha$ chosen so that $F_-(u)$ has zero mean value, according to (5.6). Determination of the constants is then completed by applying conditions (5.7), which in terms of $F_\pm$ are

$$\begin{bmatrix} 1-\beta \\ \frac{1}{3}(\beta-\alpha) \end{bmatrix} = \frac{1}{2\pi}\int_0^a \begin{bmatrix} F_+(u) \\ F_-(u) \end{bmatrix} \ln\left(\frac{1+u}{1-u}\right)\mathrm{d}u. \tag{5.9}$$

When the integral equations (5.8) are solved by the method described in the Appendix and the constants determined by the conditions (5.6) and (5.9), it is found that the variation with a of the dimensionless drag coefficient β is very similar to that of D for the same disk moving axisymmetrically. The actual forces in the two cases are $32U\mu\beta/3$ and $16U\mu D$. Values of β are displayed in figure 4; α exceeds β but the difference is less than 1% for $a < 0.97$. The second curve enables comparison to be made with the drag experienced by an open torus that translates along a transverse axis and has the same size hole, i.e. radii ratio $(1-a)/(1+a)$. The force $6\pi\mu U f_x$ calculated by Goren & O'Neill (1980) is compared with $32U\mu\beta/3$ by plotting values of $9\pi f_x/16$ which, of course, must be larger, for each a, and approach zero as $a \to 1$. However 99/101 is the largest value of a for which f_x was computed.

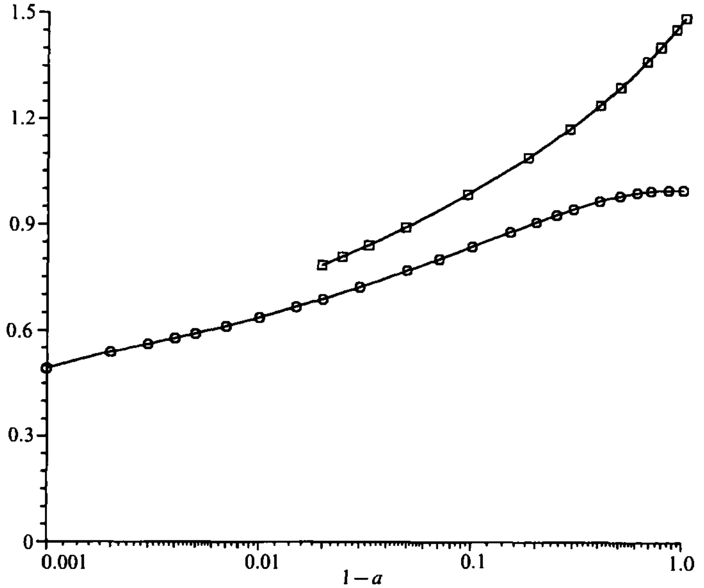

FIGURE 4. The dimensionless drag coefficient β (○) for the sideways motion of an annular disk of radii a and 1 in infinite fluid. Also shown are values of the corresponding coefficient $9\pi f_x/16$ computed by Goren & O'Neill (□) for a torus with radii ratio $(1-a)/(1+a)$. The limit values as $a \to 0$ are 1 and 1.4904 respectively.

The tangential velocity in the hole, relative to the disk, is given, from (5.5a, b) with use of (5.7), by

$$\frac{U}{\pi}(1,0)\int_\rho^a \frac{t^{-1}F_+(t)}{(t^2-\rho^2)^{\frac{1}{2}}}\,\mathrm{d}t - \frac{2U}{\pi\rho^2}(\cos 2\theta, \sin 2\theta)\int_\rho^a F_-(t)\left[\frac{t}{(t^2-\rho^2)^{\frac{1}{2}}}+\frac{(t^2-\rho^2)^{\frac{1}{2}}}{t}\right]\mathrm{d}t$$

and evidently must have a pair of symmetrically placed point vortices on the y-axis ($\theta = \frac{1}{2}\pi$).

This work was supported by NSF Grant DMS-8714694.

Appendix

The numerical solution of the systems of integral equations, (2.15), (3.9), (4.23) and (5.8) was obtained by using El-Gendi's (1969) method based on the Clenshaw–Curtis quadrature scheme to express the integral of a typical smooth function $g(u)$, defined in (0, 1), in terms of the set $[g_j; j = 0, 1, \ldots, 2N\}$ of approximate values of $g(u)$ at $u = \frac{1}{2}[1-\cos(j\pi/2N)] = \sin^2(j\pi/4N)$. Thus

$$\int_0^1 g(u)\,\mathrm{d}u \approx \sum_{j=0}^{N}{}'' b_j(g_j+g_{2N-j}), \tag{A 1}$$

where $$b_j = \frac{1}{N}\sum_{r=0}^{N}{}'' \frac{1}{1-4r^2}T_{2r}[\cos(j\pi/2N)] \quad (0 \leqslant j \leqslant N),$$

T_m denotes the mth Chebyshev polynomial and the attachment of double primes to a summation symbol indicates that the first and last terms are to be halved. In this way, the Fredholm integral equations, (2.15), (3.9), the connected pair (4.23) and the disjoint pair (5.8), are converted to sets of $2N$, $2N+1$ or $4N$ simultaneous equations for which an IMSL routine is available.

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